[2] Random tensor theory.

- The effect of Wick ordering is to eliminate the frequency "pairings":

  - $:|u|^2u: = |u|^2u - \underline{2\lambda_N \cdot u}$

$$\widehat{|u|^2u}(k) = \sum_{k_1-k_2+k_3=k} \hat{u}(k_1)\overline{\hat{u}(k_2)}\,\hat{u}(k_3).$$

  - We say a "pairing" if $k_1=k_2$ or $k_2=k_3$
  - If $k_1=k_2$, we get

$$\hat{u}(k)\cdot\sum_{k_1}|\hat{u}(k_1)|^2$$

$$= \widehat{\|u\|_{L^2}^2\cdot u}(k) \sim \lambda_N\cdot\hat{u}(k).$$

Note that $\|u\|_{L^2}^2 \sim \mathbb{E}\|u\|_{L^2}^2 = \lambda_N$.

- Because of this, we will replace the Wick-ordered nonlinearity

$$:|u|^{p-1}u: \rightsquigarrow \mathcal{N}(u,\cdots,u)$$

where

$$\boxed{\mathcal{N}(u,u,\cdots u)_k} = \sum_{\substack{k_1-k_2+\dots+k_p=k \\ \text{"no pairing"}}} u_{k_1}\overline{u_{k_2}}\cdots u_{k_p}.$$

we understand $u_k = \hat{u}(k)$

- Recall that
  - Schrödinger fundamental solution has no smoothing.
  - But we can gain a factor $\frac{1}{\langle \Omega \rangle}$, where $\underbrace{\Omega = |k|^2 - |k_1|^2 + \dots - |k_p|^2}_{\text{"Resonance factor"}}$
  - Because of this, we can formally restrict to the set "$\Omega = 0$" or "$\Omega =$ Constant"
  - In real problem, we need to introduce the $X^{s,b}$ norm of Bourgain.
- We now introduce the simplified model
  - (NLS) $\Rightarrow$ $u(t) = e^{it\Delta} f(\omega) - i \int_0^t e^{i(t-s)\Delta} \mathcal{N}(u(s), \dots u(s))\, ds$.
  - If we define $a_k(t) = e^{it|k|^2} u_k(t)$ then

$$a_k(t) = f_k - i \sum_{\substack{k_1 - \dots + k_p = k \\ \text{"no pairing"}}} \int_0^t e^{is\Omega} a_{k_1}(s) \overline{a_{k_2}(s)} \cdots a_{k_p}(s)\, ds.$$

$$\Omega = |k|^2 - |k_1|^2 + \dots - |k_p|^2.$$

  - We may restrict to $\Omega = 0$. then assume $a_k(t) = a_k$. then we get the simplified model:

$$\boxed{a_k = f_k - i \sum_{\substack{k_1 - \dots + k_p = k \\ \boxed{\Omega = 0} \text{ "no pairing"}}} a_{k_1} \overline{a_{k_2}} \cdots a_{k_p},} \quad f_k = \frac{g_k(\omega)}{\langle k \rangle^{\alpha}}$$

$\hookrightarrow$ (Model)

- We introduce the notation

$$\mathcal{M}(u^1, \cdots u^p) = -i \sum_{\substack{k_1 + \cdots + k_p = k \\ \Omega = 0}} (u^1)_{k_1} \cdots (u^p)_{k_p}.$$

then (Model) can be written as

$$a_k = \underline{f_k} + \mathcal{M}(a, \cdots a)_k.$$

- Let a_N be the solution to

$$(a_N)_k = (f_N)_k + \mathcal{M}(a_N, \cdots a_N)_k. \quad (f_N)_k = \begin{cases} f_k, & \text{if } |k| \le N \\ 0, & \text{otherwise.} \end{cases}$$

and define $y_N = a_N - a_{N/2}$, then y_N solves

$$\boxed{\begin{aligned} &\underline{(y_N)_k} = \underline{(F_N)_k} + \sum_{\max(N_1 \cdots N_p) = N} \underline{\mathcal{M}(y_{N_1}, \cdots, y_{N_p})}. \\ &(F_N)_k = \begin{cases} f_k, & \text{if } \frac{N}{2} < |k| \le N \\ 0, & \text{otherwise} \end{cases} \qquad (*) \end{aligned}}$$

- This is not rigorous. but we will assume $(a_N)_k$ and $(y_N)_k$ is supported in $|k| \le N$.

- <u>Idea</u>: we will iterate with the equation $(*)$. But will stop iterating when we reach "low frequency" — frequency $< N^\delta$.

- Terms we get in the iteration: ($p=3$):

$$F_N,\ \mathcal{M}(F_{N_1}, F_{N_2}, F_{N_3}),\ \mathcal{M}(F_{N_1}, F_{N_2}, y_{N^\delta}).$$

$$\boxed{\mathcal{M}(F_{N_1}, y_{N^\delta}, \mathcal{M}(F_{N_2}, F_{N_3}, \boxed{y_{N^\delta}}))}$$

- Goal is to write an ansatz that includes all these terms, estimate them in suitable norms and control the remainder.

- Note that all these terms have the form

$$(\text{term})_k = \sum_{k_1 \cdots k_r} \boxed{h_{kk_1\cdots k_r}} \cdot (F_{N_1})^{\pm}_{k_1} (F_{N_2})^{\pm}_{k_2} \cdots (F_{N_r})^{\pm}_{k_r} \qquad \text{(Terms)}$$

where $z^+ = z$, $z^- = \bar{z}$.

  - Note that $N_j > N^\delta$ moreover $h = h_{kk_1\cdots k_r}$ is Borel function of $\boxed{(g_k)_{|k| \le N^\delta}}$, this means that $h_{kk_1\cdots k_r}$ is independent with all the $(F_{N_j})_{k_j}$, $1 \le j \le r$.
  - In the terms we assume that $\max(N_1 \cdots N_r) = N$, and $N_j > N^\delta$. Actually the h depends on $(N_1, \cdots N_r)$, but we omit for simplicity

- In the terms we can assume there is no pairing in $(k_1 \cdots k_r)$. In fact, if we have $k_1 = k_2$, then we can rewrite this term as

$$(\text{term})_k = \sum_{k_1} \sum_{k_3 \cdots k_r} h_{k k_1 k_1 k_3 \cdots k_r} \cdot \underbrace{|(F_{N_1})_{k_1}|^2}_{\substack{\downarrow \\ \langle k_1 \rangle^{-2\alpha}}} (F_{N_3})^{\pm}_{k_3} \cdots (F_{N_r})^{\pm}_{k_r}$$

$$= \underline{\sum_{k_3 \cdots k_r} \tilde{h}_{k k_3 \cdots k_r} (F_{N_3})^{\pm}_{k_3} \cdots (F_{N_r})^{\pm}_{k_r}}$$

$$\text{where } \tilde{h}_{k k_3 \cdots k_r} = \sum_{k_1} \langle k_1 \rangle^{-2\alpha} h_{k k_1 k_1 k_3 \cdots k_r}$$

- Finally, in the terms we always have

$$\underline{k = \pm k_1 \pm \cdots \pm k_r + O(N^{\delta})}$$

- As an easy example, if we consider

$$\underline{\mathbb{E} \sum_k |(\text{Term})_k|^2} = \sum_k \mathbb{E} \Big| \sum_{k_1 \cdots k_r} \underline{h_{k k_1 \cdots k_r}} \cdot (F_{N_1})^{\pm}_{k_1} \cdots (F_{N_r})^{\pm}_{k_r} \Big|^2$$

$$= \sum_{(k, k_1 \cdots k_r)} \langle k_1 \rangle^{-2\alpha} \cdots \langle k_r \rangle^{-2\alpha} \cdot \mathbb{E} \, | h_{k k_1 \cdots k_r} |^2$$

$$= \underline{\mathbb{E} \, \| \langle k_1 \rangle^{-\alpha} \cdots \langle k_r \rangle^{-\alpha} h_{k k_1 \cdots k_r} \|^2_{\ell^2}}$$

- First, we know that the property of terms is determined by property of the tensors h.

  Second: the ℓ^2 norm of h will play a key role in the analysis.

- Full ansatz will look like

$$\underline{(y_N)_k = \sum_r \sum_{(N_1 \cdots N_r)} \Big(\sum_{k_1 \cdots k_r} \underbrace{h_{kk_1 \cdots k_r}} \prod_{i=1}^{r} (F_{N_i})^{\overset{\xi_i}{\nearrow}\pm}_{k_i} \Big)} + \underbrace{(z_N)_k}_{\text{remainder}}.$$

$\boxed{2.1}$ Random tensors and their algebra.

- Given a finite index set A, we write $k_A = (k_j)_{j \in A}$. We will denote tensors $h = h_{k_A}$. We will also write $h = h_{kk_A}$, or $h = h_{abc}$, ...

- Ansatz.

$$(y_N)_k = \sum_A \sum_{(N_i : i \in A)} \Big(\sum_{k_A} h_{kk_A} \cdot \prod_{i \in A} (F_{N_i})^{\pm}_{k_i} \Big) + (z_N)_k.$$

- Consider nonlinearity

$$\mathcal{M}(u^1 \cdots u^p)_k = \sum_{\substack{k_1 - \cdots + k_p = k \\ \Omega = 0}} (u^1)_{k_1} \cdots (u^p)_{k_p}.$$

$$= \underline{\sum_{k_1 \cdots k_p} (h^b)_{kk_1 \cdots k_p} \cdot (u^1)_{k_1} \cdots (u^p)_{k_p}}$$

where $(h^b)_{kk_1 \cdots k_p} = \begin{cases} 1, & \text{if } k_1 - \cdots + k_p = k \text{ and } \Omega = 0 \\ 0, & \text{otherwise.} \end{cases}$ "base tensor"

- If we plug in a tensor term for each of $u^1 \cdots u^p$, we get

$$(\text{Term})_k = \sum_{k_1 \cdots k_p} (h^b)_{k k_1 \cdots k_p} \cdot \prod_{j=1}^{p} \left(\sum_{k_{A_j}} h^j_{k_j k_{A_j}} \cdot \prod_{i \in A_j} (F_{\nu_i})_{k_i} \right)$$

$$= \sum_{k_{A_1} \cdots k_{A_p}} \underbrace{\left(\sum_{k_1 \cdots k_p} (h^b)_{k k_1 \cdots k_p} \cdot \prod_{j=1}^{p} h^j_{k_j k_{A_j}} \right)} \underbrace{\prod_{j=1}^{p} \prod_{i \in A_j} (F_{\nu_i})_{k_i}}$$

Moreover, we need to identify all frequency pairings across the k_{A_j}. Suppose B is the set of unpaired indices, then we have

$$(\text{Term})_k = \sum_{k_B} H_{k k_B} \cdot \prod_{i \in B} (F_{\nu_i})_{k_i}$$

where

$$\underline{\underline{H_{k k_B}}} = \sum_{k_1 \cdots k_p} \sum_{(\text{paired})} (h^b)_{k k_1 \cdots k_p} \cdot \prod_{j=1}^{p} h^j_{k_j k_{A_j}} \cdot \boxed{\prod_{(\text{paired})} \langle k_i \rangle^{-\alpha}}$$

- This is an example of what we may call "semi-produce" of tensors, where we take tensor product, then assign a set of repeated indices and sum over them.

  - Example: if we have X_{abc} and Y_{def} then we can get:

$$\begin{cases} Z_{abcdef} = X_{abc} \cdot Y_{def} \\ Z_{bcef} = \sum_a X_{abc} Y_{aef} \\ Z_{cf} = \sum_{a,b} X_{abc} Y_{abf} \end{cases}$$

- Therefore, the output tensor = the semi product of the input tensors with the base tensor h^b (under some choice of pairings). called $\boxed{\text{merging}}$

- Note that in the terms, we require that:

$$\begin{cases} (1) \text{ the } h \text{ is a Borel function of } (g_k)_{|k|<N^\delta}; \\ (2) \text{ each } N_i \geq N^\delta. \text{ So we have independence.} \end{cases}$$

If we consider

$$\mu(y_{N_1}, y_{N_2}, \cdots, y_{N_p})$$

y_{N_j} replaced by tensor term.

$$\sum_{k_{A_j}} h_{k_j k_{A_j}} \prod_{i \in A_j} (F_{N_i})_{k_i}$$

this $N_i \geq N_j^\delta$

however we need $N_i \geq N^\delta$.

- what we do: $\underbrace{\text{let } B_j = \{ i \in A_j : N_i \geq N^\delta \}}$ we will rewrite

$$\sum_{k_{A_j}} h_{k_j k_{A_j}} \cdot \prod_{i \in A_j} (F_{N_i})^{\pm}_{k_i} = \sum_{k_{B_j}} \tilde{h}_{k_j k_{B_j}} \cdot \prod_{i \in B_j} (F_{N_i})^{\pm}_{k_i}$$

$$\text{where } \boxed{\tilde{h}_{k_j k_{B_j}} = \sum_{k_{A_j \setminus B_j}} h_{k_j k_{A_j}} \cdot \prod_{i \in A_j \setminus B_j} (F_{N_i})^{\pm}_{k_i}.}$$

then $F_{N_i} \geq N^\delta$, $\forall i \in B_j$

and $\tilde{h}_{k_j k_{B_j}}$ is still a Borel function of $(g_k)_{|k|<N^\delta}$

- This operation, we call $\boxed{\text{trimming}}$

- In general.

$$\text{Output tensor} = \text{Merge}(h^b \cdot \text{Trim}(\text{Input tensors}))$$

$\boxed{2.2}$ Tensor norms and estimates

$\boxed{\text{Definition}}$. Let (B, C) is partition of A. Then we define

$$\|h_{k_A}\|^2_{k_B \to k_C} = \sup\left\{ \sum_{k_C} \Big| \sum_{k_B} h_{k_A} \cdot y_{k_B} \Big|^2 : \sum_{k_B} |y_{k_B}|^2 = 1 \right\}$$

this is the operator norm when h is viewed as an operator from functions of k_B to functions of k_C

- By duality: $\|h\|_{k_B \to k_C} = \|h\|_{k_C \to k_B}$
- If $B = \phi$ or $C = \phi$ we get $\|h\|^2_{k_A} = \sum_{k_A} |h_{k_A}|^2$ and $\|h\|_{k_B \to k_C} \le \|h\|_{k_A}$.
- if $h = h_{bc}$ then $\|h\|_{b \to c}$ = operator norm of the matrix h

  $\|h\|_{bc}$ = Hilbert-Schmidt norm of h.

$\boxed{\text{Lemma}}$ Let $h^1_{k_A}$ and $h^2_{k_B}$ be two tensors and consider the semi product.

$$H_{k_D} = \sum_{k_C} h^1_{k_A} \cdot h^2_{k_B} \quad \text{where } D = A \triangle B. \quad C = A \cap B.$$

(Example: if h^1_{abc} and h^2_{ade} then $H_{bcde} = \sum_a h^1_{abc} h^2_{ade}$)

Now let (X, Y) be a partition of D, then

$$\underline{\|H\|_{k_X\to k_Y} \le \|h^1\|_{k_{(X\cup B)\cap A}\to k_{Y\cap A}}\cdot\|h^2\|_{k_{X\cap B}\to k_{(Y\cup A)\cap B}}.}$$

(Example: as above, then $\|H\|_{bd\to ce} \le \|h^1\|_{ab\to c}\cdot\|h^2\|_{d\to ae}$.

$\boxed{\text{Proof}}$ Denote $k_{X\cap A}=a$, $k_{Y\cap A}=b$. $k_{X\cap B}=c$. $k_{Y\cap B}=d$, $k_{A\cap B}=e$.

then $h^1=h^1_{abe}$, $h^2=h^2_{cde}$, and

$$H=H_{abcd}=\sum_e h^1_{abe}h^2_{cde} \qquad \text{we estimate } \|H\|_{ac\to bd}$$

choose $y=y_{bd}$ such that $\sum_{b,d}|y_{bd}|^2=1$. Then

$$\begin{aligned}
\sum_{a,c}\Big|\sum_{b,d}H_{abcd}\,y_{bd}\Big|^2 &= \sum_{a,c}\Big|\sum_{b,d,e}h^1_{abe}h^2_{cde}\,y_{bd}\Big|^2\\
&= \sum_a\sum_c\Big|\sum_{d,e}h^2_{cde}\Big(\sum_b h^1_{abe}y_{bd}\Big)\Big|^2\\
&\le \sum_a \|h^2\|^2_{c\to de}\cdot\sum_{d,e}\Big|\sum_b h^1_{abe}y_{bd}\Big|^2\\
&= \|h^2\|^2_{c\to de}\cdot\sum_d\sum_{a,e}\Big|\sum_b h^1_{abe}y_{bd}\Big|^2\\
&\le \|h^2\|^2_{c\to de}\cdot\|h^1\|^2_{ae\to b}\cdot\underbrace{\sum_d\sum_b|y_{bd}|^2}_{=1}
\end{aligned}$$

Then we get

$$\|H\|_{ac\to bd}\le\|h^1\|_{ae\to b}\cdot\|h^2\|_{c\to de}. \qquad \#.$$

$\boxed{\text{Corollary}}$ Let $h^j=h^j_{k_{A_j}}$ be tensors, and H_D be the semi-product with some pairings.

Let (X,Y) be a partition of D. then

$$\|H\|_{k_X\to k_Y}\le\prod_j\|h^j\|_{k_{B_j}\to k_{C_j}}, \quad \text{where for each } j:$$

$$B_j = \{ i \in A_j : i \in X, \text{ or is paired with something in } A_{j'},\ j' > j \}$$

$$C_j = \{ i \in A_j : i \in Y, \text{ or is paired with something in } A_{j'},\ j' < j \}.$$

<u>Example:</u> 2f

$$H_{cugv} = \sum_{a,b,d,e,f} h^1_{abc} \cdot h^2_{adeu} \cdot h^3_{bef} \cdot h^4_{dfgv}$$

then $\underline{\|H\|_{cv \to gu} \le \|h^1\|_{abc} \cdot \|h^2\|_{de \to ua} \cdot \|h^3\|_{f \to be} \cdot \|h^4\|_{v \to dfg}}$

<u>proof</u>

- $H_{cugv} = \sum_{a,b} h^1_{abc} \cdot (H')_{abugv}, \quad (H')_{abugv} = \sum_{d,e,f} h^2_{adeu} h^3_{bef} h^4_{dfgv}$

$$\Longrightarrow \|H\|_{cv \to gu} \le \|h^1\|_{abc} \cdot \|H'\|_{v \to abgu}$$

- $(H')_{abugv} = \sum_{d,e} h^2_{adeu} \cdot (H'')_{bdegv}, \quad (H'')_{bdegv} = \sum_f h^3_{bef} h^4_{dfgv}$

$$\Longrightarrow \|H'\|_{v \to abgu} \le \|h^2\|_{de \to au} \cdot \|H''\|_{v \to bdeg}$$

- $(H'')_{bdegv} = \sum_f h^3_{bef} h^4_{dfgv}$

$$\Longrightarrow \|H''\|_{v \to bdeg} \le \|h^3\|_{f \to be} \cdot \|h^4\|_{v \to dfg}. \qquad \#$$

$\boxed{\text{Lemma 2}}$ Let $h = h_{bck_A}$ be a tensor, and define

$$H = H_{bc} = \sum_{k_A} h_{bck_A} \cdot \prod_{i \in A} g_{k_i}^{\pm}$$

Assume the followings:

(1) the tensor h is independent with the g_{k_i}'s involved.

(2) In the support of h, there is no pairing in k_A

(3) The number of choices for (b, c, k_A) is at most $N^{O(1)}$

Then, after removing a set of probability $\leq e^{-N^\theta}$ (for some $\theta > 0$), we have

$$\|H\|_{b \to c} \leq N^{\varepsilon} \cdot \sup_{(B,C)} \|h\|_{bk_B \to ck_C}$$

for any $\varepsilon > 0$, where sup is taken over all partitions (B, C) of A.

$\boxed{\text{Proof}}$ We apply a high order TT^* argument. First we need a hypercontractivity estimate:

- If $F(\omega) = \sum_{k_1 \cdots k_q} a_{k_1 \cdots k_q} \cdot g_{k_1}^{\pm} \cdots g_{k_q}^{\pm}$

then $\mathbb{P}\left(|F(\omega)| > A \cdot \left(\sum_{k_1 \cdots k_q} |a_{k_1 \cdots k_q}|^2 \right)^{\frac{1}{2}} \right) < C \cdot e^{-A^{\frac{2}{q}}}$

Now, $H = H_{bc}$ can be viewed as the kernel of an operator T, which is a linear operator from the space of functions of c to the space of functions of b.

$$T: \ell_c^2 \to \ell_b^2 . \ (Tz)_b = \sum_c H_{bc} z_c$$

$$T^*: \ell_b^2 \to \ell_c^2 \ (T^* y)_c = \sum_b \overline{H_{bc}} y_b .$$

$$TT^*: \ell_b^2 \to \ell_b^2 . \ (TT^* y)_b = \sum_{b'} \Big(\sum_c H_{bc} \overline{H_{b'c}} \Big) y_{b'}$$

......

In general, we will consider the operator $T_n = \begin{cases} (TT^*)^m, & \text{if } n = 2m \\ (TT^*)^m T, & \text{if } n = 2m+1 . \end{cases}$

We will prove that for each n, the kernel (H_n) of T_n has the form

- If $n = 2m$: $(H_n)_{bb'} = \sum_{k_{A_n}} (R_n)_{bb'k_{A_n}} \prod_{i \in A_n} g_{k_i}^{\pm}$

- If $n = 2m+1$: $(H_n)_{bc} = \sum_{k_{A_n}} (R_n)_{bck_{A_n}} \prod_{i \in A_n} g_{k_i}^{\pm}$

In the above we have

$$\|R_n\|_{bb'k_{A_n}} \ (\text{or } \|R_n\|_{bck_{A_n}}) \le D^{n-1} \cdot \|h\|_{bck_A}$$

where

$$D = \sup_{(B,C)} \|h\|_{bk_B \to ck_C}$$

We induct in n. When $n = 1$, we may choose $A_1 = A$. Suppose this is true for n, where n is even, then $T_{n+1} = T_n \cdot T$. thus

$$(H_{n+1})_{bc} = \sum_{b'} (H_n)_{bb'} \cdot H_{b'c}$$

$$= \sum_{b'} \sum_{k_{A_n}} \sum_{k_A} (R_n)_{bb'k_{A_n}} \cdot h_{b'ck_A} \cdot \prod_{i \in A_n \cup A} (g_{k_i})^{\pm} .$$

Then we identify the pairings in $g_{k_i} : i \in A_n \cup A$, and let the set of unpaired indices be A_{n+1}, then

$$(H_{n+1})_{bc} = \sum_{k_{A_{n+1}}} (R_{n+1})_{bck_{A_{n+1}}} \cdot \prod_{i \in A_{n+1}} (g_{k_i})^{\pm}$$

where

$$(R_{n+1})_{bck_{A_{n+1}}} = \sum_{(\text{paired indices})} \sum_{b'} (R_n)_{bb'k_{A_n}} \cdot \underline{h_{b'ck_A}}$$

is a semi-product of R_n with h. By Lemma:

$$\|R_{n+1}\|_{bck_{A_{n+1}}} \leq \|R_n\|_{bb'k_{A_n}} \cdot \|h\|_{\underline{ck_C} \to \underline{bk_B}} \leq \underline{D} \cdot \underline{\|R_n\|_{bb'k_{A_n}}}.$$

for some partition (B, C) of A. This completes the inductive proof.

Now, choose $n = 2m$, we have

$$\|T\|^{4m}_{\ell^2 \to \ell^2} = \|(TT^*)^m\|^2_{\ell^2 \to \ell^2} = \|T_{2m}\|^2_{\ell^2 \to \ell^2}$$

$$\leq \sum_{b,b'} |(H_{2m})_{bb'}|^2.$$

We know that, up to a set of probability $\leq e^{-N^\theta}$, we have

$$|(H_{2m})_{bb'}|^2 \leq \sum_{k_A} |(R_{2m})_{bb'k_A}|^2 \cdot N^\varepsilon \quad \text{for all } (b,b').$$

Then $\|T\|^{4m}_{\ell^2 \to \ell^2} \leq \|R_{2m}\|^2_{bb'k_A} \leq D^{4m-2} \cdot \|h\|^2_{bck_A}$

Finally, note that $\|h\|_{bck_A} \leq N^{O(1)} \cdot D$, then

$$\|H\|_{b\to c} = \|T\|_{\ell^2\to\ell^2} \le D\cdot\left(\frac{\|h\|_{bck_A}}{D}\right)^{\frac{1}{2m}}$$

$$\le D\cdot N^{\frac{O(1)}{2m}}$$

choose m large. we get this $\le D\cdot N^{\varepsilon}$.

#

$\boxed{2.3}$ Local well-posedness for (NLS).

- Recall the model

$$(y_N)_k = (F_N)_k + \sum_{\max(N_1\cdots N_p)=N} \mathcal{M}(y_{N_1}\cdots y_{N_p})_k$$

We introduce the ansatz

$$(y_N)_k = \underbrace{\sum_S\left(\sum_{k_A}\boxed{h_{kk_A}}\prod_{i\in A}(F_{N_i})^{\pm}_{k_i}\right)} + z_N(k).$$

where S is a set A, together with N_i for each $i\in A$. such that $N_i\ge N^{\delta}$, and $\max_{i\in A} N_i = N$. Moreover we require that $|A|\le D$. (where D is a large constant).

- As defined before. if we have input tensors $h^{(j)}_{k_jk_{A_j}}$, $1\le j\le p$. then the output tensor $h = h_{kk_A} = \boxed{\text{Merge}(h^b, \text{Trim}(h^{(j)})_{1\le j\le p})}$

Using this we can inductively define the tensors that appear in the ansatz, and we need to inductively prove the bounds of these tensors in the norms of our choice.

- Recall $\alpha = s+\frac{d}{2}$. $s>s_{pr} = -\frac{1}{p-1}$. Define $\alpha_1 = -\frac{1}{p-1}+\frac{d}{2}$. then choose $\alpha_0 = \frac{1}{2}(\alpha+\alpha_1)$. then $\alpha_1<\alpha_0<\alpha$. We pose the following estimates on h_{kk_A}:

(1) In the support of h_{kk_A} we have

$$k = \sum_{i\in A} (\underset{\sim}{\pm k_i}) + O(N^\delta).$$

(2) For any partition (P,Q) of A we have

$$\underline{\|h\|_{kk_P \to k_Q} \le \underset{\sim\sim\sim}{\prod_{i\in A} N_i^{\alpha_0}}} \cdot \begin{cases} (\max_{i\in Q} N_i)^{-\alpha_0} & \text{if } Q \ne \phi \\ 1 & \text{if } Q = \phi. \end{cases} \quad (*)$$

(needs correction factors in real case).

- To finish the inductive proof we need to show two things:

  (1) If h satisfies $(*)$ then $\mathrm{Trim}(h)$ also satisfies $(*)$;

  (2) If $h_1 \cdots h_p$ satisfy $(*)$ then $\mathrm{Merge}(h^b, h_1, \cdots h_p)$ also satisfies $(*)$.

- First look at (1). Let $h = h_{kk_A}$ and $B = \{ i \in A : N_i \ge N^\delta \}$. Notice that there is some M ($M \le N$) such that $N_i \ge M^\delta$ ($\forall i \in A$) and h is a Borel function of $(g_k)_{|k|<M^\delta}$. Then by definition

$$\mathrm{Trim}(h) = \tilde{h}_{kk_B} = \sum_{k_{A\setminus B}} h_{kk_A} \cdot \prod_{i\in A\setminus B} g_{k_i}^{\pm} \underset{\sim\sim\sim}{\langle k_i \rangle^{-\alpha}}$$

$$\to \sim N_i^{-\alpha}$$

Since h is Borel function of $(g_k)_{|k|<M^\delta}$, and $N_i \ge M^\delta$ for $i \in A\setminus B$, and there is no pairing in $k_{A\setminus B}$ in this summation, we can apply the lemma and get, up to a set of probability $\le e^{-N^\theta}$, that

$$\|\mathrm{Trim}(h)\|_{kk_P \to k_Q} \le N^\varepsilon \cdot \prod_{i\in A\setminus B} N_i^{-\alpha} \cdot \underline{\sup_{(E,F)} \|h\|_{kk_{P\cup E} \to kk_{Q\cup F}}}$$

where (P,Q) = partition of B, and (E,F) = partition of $A\setminus B$. Now, if $Q \ne \phi$

then

$$\|\mathrm{Trim}(h)\|_{kk_P\to k_Q} \le N^{\varepsilon}\cdot \prod_{i\in A\setminus B} N_i^{-\alpha}\cdot \prod_{i\in A} N_i^{\alpha_0}\cdot \left(\max_{i\in OUT} N_i\right)^{-\alpha_0}$$

$$= \underbrace{\prod_{i\in B} N_i^{\alpha_0}\cdot\left(\max_{i\in Q} N_i\right)^{-\alpha_0}}\cdot \underbrace{N^{\varepsilon}\cdot\prod_{i\in A\setminus B} N_i^{\alpha_0-\alpha}}_{\le 1}.$$

- Finally we consider the merging estimates. We need 2 ingredients:

(1) an estimate for norms of h^b, which will follow from suitable counting estimates:

(2) a selection algorithm of rearranging $h_1 \cdots h_p$ in applying the multilinear lemma to get the optimal bound.

- Recall that

$$(h^b)_{kk_1\cdots k_p} = \begin{cases} 1, \text{ if } \pm k_1 \pm \cdots \pm k_p \pm k = 0,\ \Omega = 0,\ \text{no pairing} \\ \\ 0, \text{ otherwise.} \end{cases}$$

$\boxed{\text{Lemma}}$ Suppose $(k_1, \cdots, k_p)$ satisfies that

$$\left\{\begin{array}{l} \pm k_1 \pm \cdots \pm k_p = \boxed{\text{Const}} \\ \pm |k_1|^2 \pm \cdots \pm |k_p|^2 = \boxed{\text{Const}} \\ |\pm k_p - m_p| \le M_p \\ |\pm k_{p-1} \pm k_p - m_{p-1}| \le M_{p-1} \\ \vdots \\ |\pm k_3 \pm \cdots \pm k_p - m_3| \le M_3 \\ |\pm k_2 \pm \cdots \pm k_p - m_2| \le M_2 \end{array}\right.$$

where $m_2, \cdots, m_p$ are constant vectors. then the number of choices for $(k_1, \cdots, k_p)$ is at most $(M_2 \cdots M_p)^{2\alpha_1+\varepsilon}$. The same is true if p is replaced by $q \le p$

$\boxed{\text{Proof}}$ We induct on p. If $p=3$. then $\boxed{2\alpha_1 = d-1}$ Consider the case where

$$\begin{cases} \cdot\ k_1 - k_2 + k_3 = k \ (\text{Const}) \\ \cdot\ |k_1|^2 - |k_2|^2 + |k_3|^2 = \Omega \ (\text{Const}) \\ \quad |k_3 - m_3| \le M_3 \longrightarrow k_3 \in B_{M_3} \\ \quad |-k_2 + k_3 - m_2| \le M_2 \longrightarrow k_1 \in B_{M_2}. \end{cases}$$

then. by algebra we get $(k-k_1)\cdot(k-k_3) = (\text{Constant})$. The first $(d-1)$ coordinates of $(k-k_1)$ and $(k-k_3)$ has at most $M_2^{d-1} \cdot M_3^{d-1}$. Suppose they are fixed. then we have that the choice for the last coordinates is $\boxed{\le (M_2 M_3)^{\varepsilon}}$ by the divisor estimate:

$$\forall A \ne 0.\ \#\{(b, c) \in \mathbb{Z}^2 : bc = A\} \le A^{\varepsilon}, \ \forall \varepsilon > 0.$$

So we get $(M_2 M_3)^{d-1+\varepsilon}$

Now suppose the result is true for $p-2$, then for p.

- Fix the values of k_1 and k_2. ($k_1 \in B_{M_2}$ and $\pm k_1 \pm k_2 \in B_{M_3}$) So we get $(M_2 M_3)^d$: for the rest. apply induction hypothesis we get

$$(\#\text{ of choices}) \le (M_2 M_3)^d \cdot (M_4 \cdots M_p)^{d - \frac{2}{p-3} + \varepsilon}.$$

- Fix the values of $(k_4, \cdots, k_p)$ we get $(M_4 \cdots M_p)^d$. Then apply $p=3$ case to (k_1, k_2, k_3). we get

$$(\#\text{ of choices}) \le (M_4 \cdots M_p)^d \cdot (M_2 M_3)^{d-1+\varepsilon}.$$

Interpolating between these two we get

$$(\#\text{ of choices}) \le (M_2 M_3)^{d-\frac{2}{p-1}+\varepsilon} (M_4 \cdots M_p)^{d-\frac{2}{p-1}+\varepsilon}.$$

$\boxed{\text{Corollary}}$ Suppose we restrice the tensor h^b to the set

$$\begin{cases} |\pm k_p| \le M_p \\ |\pm k_{p-1} \pm k_p| \le M_{p-1} \\ \vdots \\ |\pm k_2 \pm \cdots \pm k_p| \le M_2 . \end{cases}$$

then we have $\|h^b\|_{kk_X \to k_Y} \le (M_2 \cdots M_p)^{\alpha_1+\varepsilon}$, as long as $1 \in Y$.

$\boxed{\text{Proof}}$ We apply Schur's inequality

$$\|h^b\|_{kk_X \to k_Y} \le \Big(\sup_{(k,k_X)} \sum_{k_Y} |h^b|\Big)^{\frac{1}{2}} \Big(\sup_{k_Y} \sum_{(k,k_X)} |h^b|\Big)^{\frac{1}{2}}$$

By Lemma we can get

$$\sup_{(k,k_X)} \sum_{k_Y} |h^b| \le \Big(\prod_{j \in Y \setminus \{1\}} M_j\Big)^{\alpha_1+\varepsilon}$$

$$\sup_{k_Y} \sum_{(k,k_X)} |h^b| \le \Big(\prod_{j \in X} M_j\Big)^{\alpha_1+\varepsilon}$$

$$\Longrightarrow \|h^b\|_{kk_X \to k_Y} \le (M_2 \cdots M_p)^{\alpha_1+\varepsilon}. \quad \forall \varepsilon > 0.$$

- Now we start the proof of merging estimate. Recall that input tensors

$$h_j = (h_j)_{k_j k_{A_j}} \qquad \text{satifies } (*)$$

We will later rearrange h_j into $h_{(1)}, h_{(2)}, \cdots h_{(p)}$, say $h_{(j)} = (h_{(j)})_{k_{(j)} k_{A_{(j)}}}$.

- Recall that $H = H_{kk_A}$ is the semi-product of $h^b_{kk_1 \cdots k_p}$ and $h_{(1)}, \cdots h_{(p)}$. note that A = set of unpaired indices. Suppose we want to estimate

$\|H\|_{kk_P \to k_Q}$ $\quad (P, Q) = $ partition of A. $Q \neq \emptyset$

- First choose $i_1 \in Q$ such that $N_{i_1} = M_1$ is max. Suppose $i_1 \in A_{(1)}$
- Next. Consider all indices in other tensors that either $\in P$, or $\in Q$, or is paired with an element in $A_{(1)}$. Choose i_2 such that $N_{i_2} = M_2 = \max$. Suppose $i_2 \in A_{(2)}$. We also give $A_{(2)}$ (and i_2, $h_{(2)}$) a type P or Q: if $i_2 \in P$ (or Q) then it has this type: otherwise if i_2 is paired with an element of $A_{(1)}$ then it has type Q.
- We continue this process. Note each time, if $i_j \in P$ (or Q) then it has type P (or Q). if it is paired with $A_{(1)}$ then it has type Q: if it is paired with $A_{(j')}$ where $1 < j' < j$. then it has same type as $A_{(j')}$.
- Example, if $p = 5$, and

$$h_1 = (h_1)_{k_1 x c} \quad h_2 = (h_2)_{k_2 ab} \quad h_3 = (h_3)_{k_3 cde}$$

$$h_4 = (h_4)_{k_4 efg} \qquad h_5 = (h_5)_{k_5 afh}$$

where $|x| \geq |a| \geq |b| \geq \cdots \geq |h|$ and

$$H = H_{kxbdgh} = \sum_{k_1 \cdots k_5} \sum_{x.a.e.f} (h^b)_{kk_1 \cdots k_5} (h_1)_{k_1 xc} (h_2)_{k_2 ab} (h_3)_{k_3 cde} (h_4)_{k_4 efg} (h_5)_{k_5 afh}$$

assume $P = \{b, d\}$. $Q = \{x, g, h\}$ then

- $i_1 = x$. $h_{(1)} = h_1$;
- $i_2 = b$. $h_{(2)} = h_2$: type P:
- $i_3 = a$. $h_{(3)} = h_5$: type P:
- $i_4 = c$. $h_{(4)} = h_3$ type Q:
- $i_5 = e$. $h_{(5)} = h_4$ type Q.

- Now, we apply the multilinear estimate. and we
  - First choose $h_{(1)}$:
  - Then choose all $h_{(j)}$ of type Q, in the increasing order of j
  - Then choose h^b:
  - Then choose all $h_{(j)}$ of type P in the decreasing order of j

By multilinear estimate, we get a bound of form

$$\|H\|_{k k_P \to k_Q} \leq \|h^b\|_{k k_X \to k_Y} \cdot \prod_{j=1}^{p} \|h_{(j)}\|_{k_{(j)} k_{P_{(j)}} \to k_{Q_{(j)}}}$$

In the example we have

$$\|H\|_{kbd \to xgh} \leq \|h^b\|_{k k_2 k_5 \to k_1 k_3 k_4} \cdot \|h_{(1)}\|_{k_1 c \to x} \cdot \|h_{(2)}\|_{\substack{k_2 a \to b \\ (b \to k_2 a)}}$$
$$\cdot \|h_{(3)}\|_{\substack{k_5 fh \to a \\ (a \to k_5 fh)}} \cdot \|h_{(4)}\|_{k_3 de \to c} \cdot \|h_{(5)}\|_{k_4 f \to eg}$$

Note that the algorithm actually guarantees that $i_j \in Q_{(j)}$ for each $2 \leq j \leq p$.

In particular

$$\max_{i \in Q_{(j)}} N_i \geq N_{i_j} = M_j.$$

Note also that $1 \in Y$. Then we estimate

$$\prod_{(\text{paired } i)} N_i^{-\alpha} \cdot \|H\|_{kk_P \to k_Q} \le \underline{\prod_{(\text{paired } i)} N_i^{-\alpha}} \cdot \underline{\prod_{i \in A_1 \cup \cdots \cup A_p} N_i^{\alpha_0}} \cdot \prod_{j=1}^{p} \left(\max_{i \in Q_{(j)}} N_i\right)^{-\alpha_0}$$

$$\cdot \|h^b\|_{kk_X \to k_Y}$$

$$\le \prod_{i \in A} N_i^{\alpha_0} \cdot \prod_{j=2}^{p} M_j^{-\alpha_0} \cdot \underline{\|h^b\|_{kk_X \to k_Y}}$$

$$\cdot \left(\max_{i \in Q} N_i\right)^{-\alpha_0}$$

So it suffices to show that

$$\|h^b\|_{kk_X \to k_Y} \le (M_2 \cdots M_p)^{\alpha_1 + \varepsilon}.$$

This is because we may assume

$$\underline{|\pm k_{(j)} \pm k_{(j+1)} \pm \cdots \pm k_{(p)}|} \lesssim M_j + \underbrace{O(N^\delta)}_{\text{omit!}}$$

In fact, we have

$$\underline{k_{(j)}} = \underline{\sum_{i \in A_{(j)}}} \pm k_i + \underline{O(N^\delta)} \quad \text{for each } j.$$

If there is a pair in $A_{(j)}, \cdots A_{(p)}$ then these will cancel, so the terms in RHS only has the indices that is either unpaired ($\in P$ or $\in Q$), or paired with $A_{(j')}, j' < j$

By our choice, for each such i, $N_i \le M_j$. So we get

$$|\pm k_{(j)} \pm \cdots \pm k_{(p)}| \lesssim M_j + O(N^\delta)$$

- Finally, notice that the tensor term

$$\eta_k = \sum_{k_A} \underline{h_{kk_A}} \cdot \prod_{i \in A} \underline{\langle k_i \rangle^{-\alpha}} g_{k_i}^{\pm}$$

$$\to N_i^{-\alpha}$$

Then with high probability, for each k, we have

$$|\eta_k|^2 \le \sum_{k_A} |h_{kk_A}|^2 \cdot \prod_{i\in A} N_i^{-2\alpha}$$

$$\le \|h\|^2_{k\to k_A} \cdot \prod_{i\in A} N_i^{-2\alpha}$$

$$\le \prod_{i\in A} N_i^{2\alpha_0} \cdot \prod_{i\in A} N_i^{-2\alpha}$$

$$= \prod_{i\in A} N_i^{2(\alpha_0-\alpha)} \le N^{2\delta(\alpha_0-\alpha)|A|}$$

$$\Longrightarrow |\eta_k| \le N^{\delta(\alpha_0-\alpha)\cdot|A|}.$$

If $|A| \gg 1$, then $|\eta_k|$ = large negative power

$\Longrightarrow$ the term has high regularity.

If we choose cutoff at degree D, where $D\cdot\delta\cdot(\alpha-\alpha_0) \gg 1$ then the terms of degree $> D$ will have high regularity, and can be found by fixed point argument.

- Thing to look at in the future.
  - Stochastic NLS
  - Other dispersion relation
  - Critical case $(d=3, p=3)$.
  - Quasilinear case ???
  - Long-time solutions.